FINITE DIFFERENCE METHOD OF ANALYSING DISPERSION OF WASTE - WATERS FROM RECHARGE WELLS IN POROUS MEDIA

BY:
Prof. Dr. Nashat A. Ali
Prof. of Hydraulics & Water Resources, Civil Eng. Dept., Assiut University, Assiut,71516, Egypt.
Dr. Mohamed E. El-Darder and Dr. Gamal Abozeid
Lecturers, Civil Eng. Dept., Assiut University, Assiut, 71516, Egypt.

Abstract: In this paper. the dispersion of waste-waters in porous media by number of recharge wells penetrating a confined aquifer is studied by solving the differential equations through the use of finite difference technique. The mathematical model adopted comprises three non-linear differential equations; the first represents the piezometric head, the second for pore water velocity and the third one for the dispersion of waste concentration. The model has been developed to determine the distribution of the piezometric head, the pore velocity and the concentration of wastes at any position in the field of multi-recharged wells. The numerical results illustrate the dependence of the piezometric head and the relative concentration on the dimensionless time-parameter and the transmissibility-parameter. The number of recharge wells has noticeable influence on the concentration of wastes at a point in the field of interference.

KEYWORD: Aquifer, Concentration, Dispersion, Miscible, Recharge, Waste, Well.

INTRODUCTION

The artificial recharge of ground water aquifers with treated waste-waters and ground disposal of wastes are practices of great importance in management and conservation of water resource planning. The need for the safe disposal of partially treated waste-water or industrial waste products has led to the pumping of these wastes through wells into ground strata. In same cases, these strata are either isolated or in other cases are parts of the aquifers used for water supply. Such practices, however, raise concern over the associated health hazard and the need for the establishment of quality criteria for ground water body receiving the waste-waters.
Ground water pollution is usually traced back to four main origins: domestic, industrial, agricultural, and environment pollution. Careful observations [1, 2, 3, 4, 5, 6, 7] showed that the most frequent and most dangerous forms of ground water pollution are miscible with water or traces if their critical masses are small enough and the movement of traced water is the best example of the miscible displacement. Field studies on dispersion of polluted water recharge by wells have been performed by Welsch [1] and Steinbrugge et al. [8]. In their works, the chemical constituents in waste- water were transported with recharged water from the well while miscible was restrained and accumulated in voids of soil particles near the well.
In the last decades, great deal of works have been directed towards on the understanding of miscible displacement in porous media. Studies on dispersion in well recharge take place in three categories:

i) Mathematical modeling process describing the dispersion phenomena [2, 8, 10, 11, 12, 13].
ii) Laboratory experiments investigating the displacement of miscible fluids in porous media [1. 7, 8, 9].

ii) The application of dispersion formulae to field problems which is quickly gaining momentum as the urgent needs for pollution behavior prediction [1, 5, 8, 9, 14].

Recently, the dispersion of waste-water concentration was studied by number of investigators among whom were De Josselin [2] Yih [3] Abdel Sadek et al. [13] and Ali and Rshwan [15]. In these works, the problem of unsteady radial flow to well penetrating an aquifer has been mathematically formulated by number of differential equations. In some works [13, 14], the solutions of the differential equations either for the drawdown near a recharging well or dispersion of salt-water are performed by using the finite difference technique.

Herein, analytic model for the unsteady waste-water flow through porous media by number of recharging wells penetrating a homogeneous, isotropic and confined aquifer is presented. The solutions of the waste-water dispersion with the aquifer characteristics (permeability, transmissibility, grain size, porosity and pore water velocity) are obtained for the confined aquifer of infinite, horizontal extent.

THEORETICAL FORMULATION

i- Single Recharge well:

The problem of unsteady radial flow from a recharging well penetrating confined aquifer (Fig. 1) can be formulated mathematically by the following system of equations [9, 13, 15].

i-1) Piezometric head: The piezometric head at any point in the flow field is given by;

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad (1)$$

in which h = piezometric head, r = radial distance from a well, S = storage coefficient, T = transmissibility coefficient and t = time.

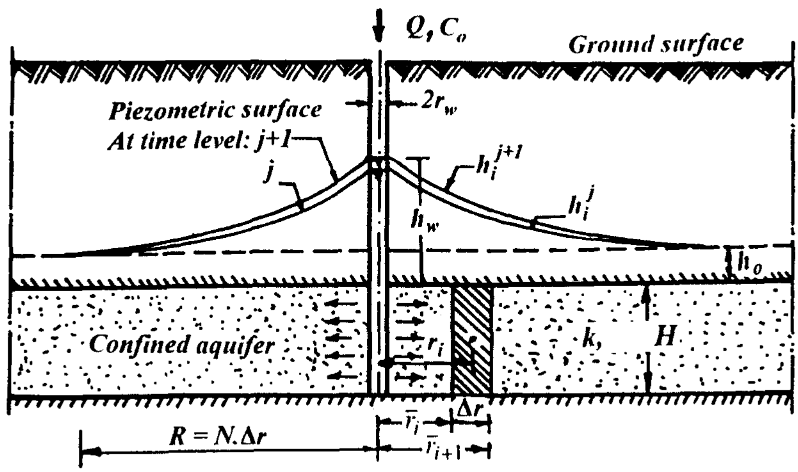


Fig. 1 Radial flow from recharge well in confined aquifer.

i-2) Equation of concentration: The dispersion of concentration of miscible of water through porous media may be expressed as [12]:

$$K_r \frac{\partial^2 C}{\partial r^2} - U\frac{\partial C}{\partial r} = \frac{\partial C}{\partial t} \qquad (2)$$

$$U = \frac{k}{\varphi}\frac{\partial h}{\partial r} \qquad (3)$$

in which ***U*** is the pore velocity that is given by;

$\boldsymbol{K_r}$ is the radial dispersion coefficient, ***k*** is the coefficient of permeability and $\boldsymbol{\varphi}$ is the soil porosity.

It is found that the radial dispersion coefficient $\boldsymbol{K_r}$ is function of both the pore velocity *U* and the median size of soil particles as [9];

$$K_r = 0.86\, U.d_s \qquad (4)$$

The solution of Eq. (1) using the finite difference method was obtained by Abdel Sadek et al. [13]. As in Fig. 2, the flow field is divided into N blocks radially from the well to the radius of influence R. Therefore, in finite difference form, Eq. (1) may be written in the form;

$$2\pi T_i^j (h_{i+1}^{j+1} - h_i^{j+1}) / \ln(r_{i+1}/r_i) + 2\pi T_i^j (h_{i-1}^{j+1} - h_i^{j+1}) / \ln(r_i / r_{i-1}) =$$
$$\pi (r_i^2 - r_{i-1}^2)(h_i^{j+1} - h_i^j) S_i^j / \Delta t \qquad (5)$$

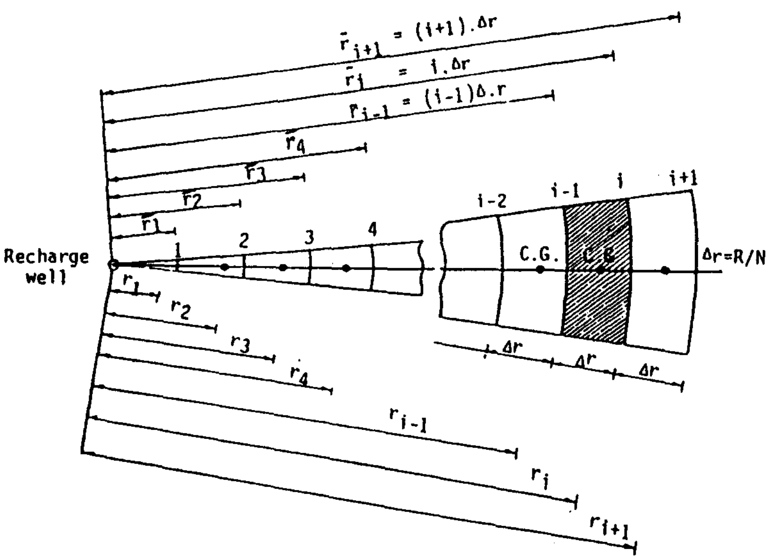


Fig. 2 Division of flow field to N blocks.

After the rearrangement of Eq. (5), it reduces to;

(6)

in which

$$A_{i+1}^{j} = a_i^j / (f_i^j - A_i^j a_{i-1}^j) \tag{7a}$$

$$B_{i+1}^{j} = (A_{i+1}^j / a_i^j)(a_{i-1}^j B_i^{j+1} + b_i^j h_i^j) \tag{7b}$$

where

$$a_i^j = 2\pi T_i^j / \ln(r_{i+1}/r_1) \tag{8a}$$

$$b_i^j = \pi (r_i^2 - r_{i-1}^2) S_i^j / \Delta t \tag{8b}$$

$$f_i^j = a_{i-1}^j + a_i^j + b_i^j \tag{8c}$$

At the first block, Eq. (6) can be written as;

$$h_1^{j+1} = A_2^j h_2^{j+1} + B_2^{j+1} \tag{9a}$$

$$A_2^j = a_1^j (a_1^j + b_1^j) \tag{9b}$$

$$B_2^{j+1} = (A_2^j / a_1^j)(h_1^j b_1^j - Q^j) \tag{9c}$$

in which T_i^j= coefficient of transmissibility = kH, (for confined aquifer of thickness H), S_i^j = storage coefficient, Δt = time increment, h_i^j = piezometric head at block i and time level j and r_i = radial distance to the center of i block which is given by;

$$r_i = \bar{r}_{i-1} + (\bar{r}_i + \bar{r}_{i-1})/2 + (\bar{r}_i + \bar{r}_{i-1})^2 / 6(\bar{r}_i + \bar{r}_{i-1}) \tag{10}$$

where $\bar{r}_i = i\Delta r = i\frac{R}{N}$

Using the boundary condition h_N^{j+1}= 0 and the obtained values of the parameters a_i^j, b_i^j, f_i^j, and A_i^j and B_i^j, the piezometric head h_N^{j+1}, h_{N-1}^{j+1},...... h_1^{j+1} at all blocks in the back direction at time level $j+1$ can be determined.

Using the finite difference method, Eq. (3) for pore velocity can be given in the form;

$$U_i^{j+1} = -\frac{k}{\varphi}(h_i^{j+1} - h_{i-1}^{j+1}) / \Delta r \tag{11}$$

In similar way, the solution of Eq. (2) may be given as;

$$(C_i^{j+1} - C_i^j)/\Delta t = K_{r_i}^j (C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1})/\Delta r^2 - U_i^j (C_{i-1}^{j+1} - C_{i-1}^{j+1})/2\Delta r \qquad (12)$$

Equation (12) may generally be expressed as;

$$C_i^{j+1} = D_{i+1}^j \, C_{i+1}^{j+1} + F_{i+1}^{j+1} \qquad (13)$$

in which

$$D_{i+1}^j = e_i^j /(g_i^j - d_i^j D_i^j) \qquad (14a)$$

where

$$F_{i+1}^j = (D_{i+1}^j / e_1^j)[d_i^j F_i^{j+1} + C_i^j (g_i^j - 2)] \qquad (14b)$$

$$d_i^j = 1.0 + (y_i^j Z)/2 \qquad (15a)$$

$$e_i^j = 1.0 - (y_i^j Z)/2 \qquad (15b)$$

$$g_i^j = (Z^2 / X_i^j) + 2 \qquad (15c)$$

and

$$Z = \Delta r / R, \qquad X_i^j = \Delta t K_{r_i}^j /(\varphi R^2) \;\; \text{and} \qquad y_i^j = U_i^j R / K_{r_i}^j$$

For numerical solution of Eq. (13), the parameters d_i^j, e_i^j and g_i^j should firstly be determined from the previous values of $K_{r_i}^j$ and U_i^j. Then the coefficients D_i^j and F_i^j can be obtained using the lower boundary condition C_N^{j+1}=0 and hence the concentrations C_N^{j+1}, C_{N-1}^{j+1}, ... C_i^{j+1}can be calculated at all nodes.

ii- Multi – Recharge Wells:

The piezometric head at a point in the field of interference caused by several wells penetrating a confined aquifer (Fig. 3) is equaling to the sum of the piezometric heads caused by each well individually. Thus;

$$h_A = h_1 + h_2 + \cdots\cdots\cdots + h_{n_w} \qquad (16)$$

in which n_w = number of interfered wells.
Similarly, the concentration of wastes at point A in the field of n_w wells is;

$$C_A = (C_1 h_1 + C_2 h_2 + \cdots\cdots + C_{n_w} h_{n_w}) / h_A \tag{17}$$

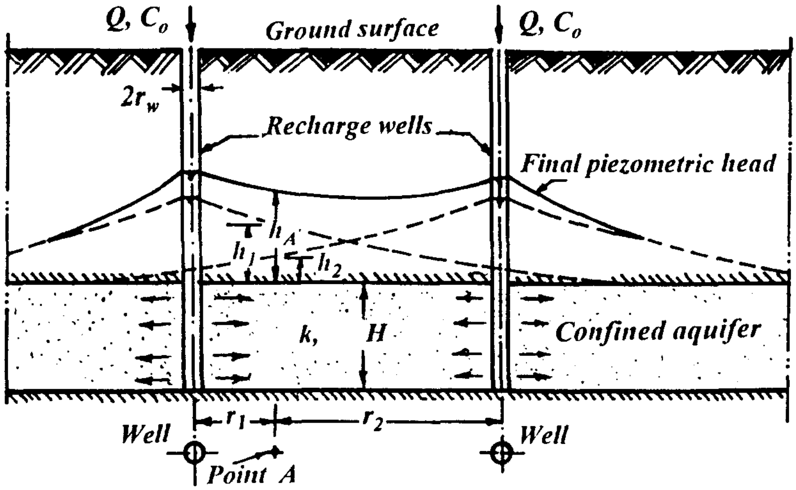


Fig. 3 Multiple recharge wells in confined aquifer.

DIMENSIONAL REASONINIG

From the foregoing, it is appeared that the concentration C_A at any point in the flow field of interfered recharge wells is function of the initial concentration C_o, the permeability coefficient k, the transmissibility coefficient T, the storage coefficient S, the porosity φ, the median size of particle d_s, recharge rate Q, radial distance from the well r_A, time t, and number of wells n_w.

Accordingly;

$$C_A = \Phi\,(C_o, T, k, S, \varphi, d_s, Q, r_A, t, n_w) \tag{18}$$

Keeping the recharge rate Q and the radial distance r_A as constants, Eq. (18) in dimensionless form becomes;

$$C_A / C_o = \Phi_1\,(T/k\,d_s, S, \varphi, k\,t/d_s, n_w) \tag{19}$$

In the model, the porosity and the storage coefficient are kept constants at an average value of 0.4 and 0.004 respectively. Therefore, Eq. (19) reduces to;

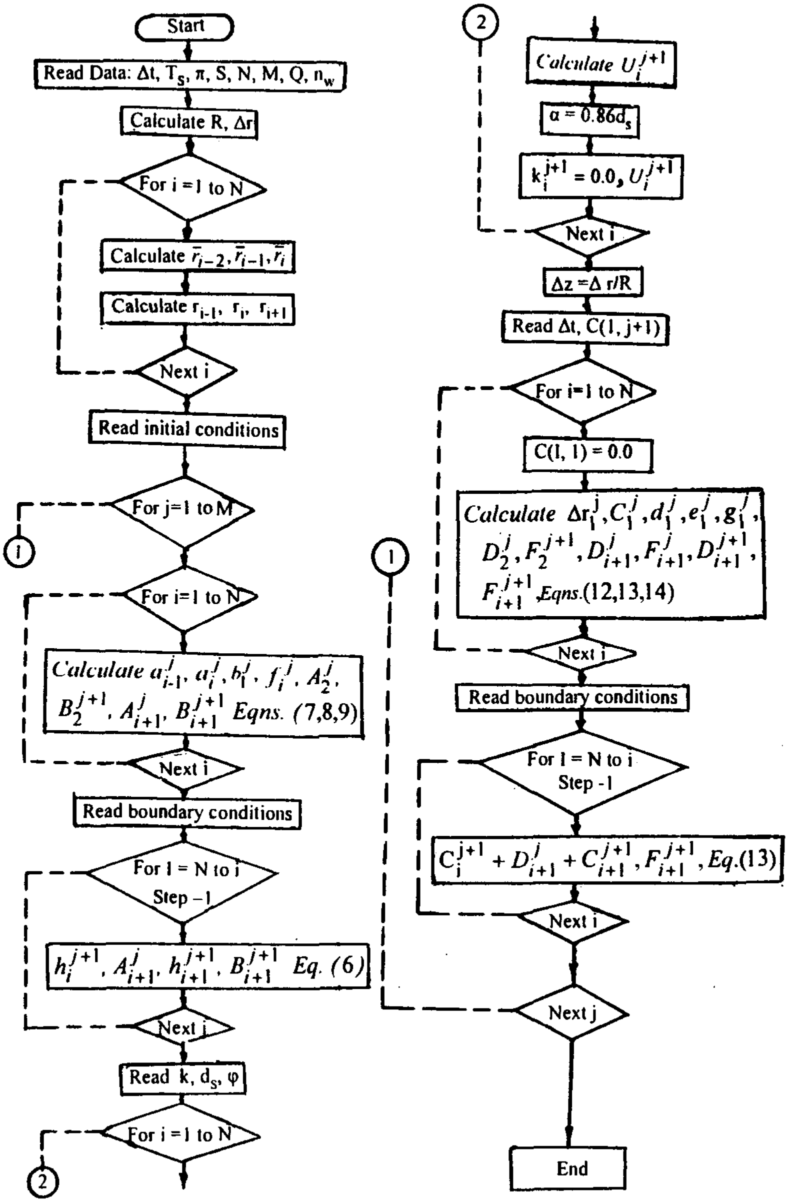
$$C_A / C_o = \Phi_2\,(T_s, K_t, n_w) \tag{20}$$

Where T_s = transmissibility- parameter = $T/k\,d_s$ and $\boldsymbol{K_t}$ = time-parameter = $k\,t/d_s$.

COMPUTER FACILITY

A computer program was written in BASIC for the use with IBM personal computer. The flow-chart of the program is shown in Fig. 4. In the program, the number of wells around a chosen position referred to A changes from 2 to 5. The wells that having recharge rates of

1000, 750, 150, 75 and 300 m^3/day are at distances 250.0, 200.0, 100.0, 50.0 and 150.0 m from the reference point A. The median size of soil particles and porosity are kept constants at 1.0 mm and 0.4 respectively [16]. The storage coefficient is assumed to be constant at 0.004 [16]. The permeability coefficient is changed from 25 to 300 m^3/day [14, 16] while the transmissibility coefficient varied from 50 to 600 m^3/day.



RESULTS AND DISCUSSIONS

In the numerical calculations, the number of divisions (N) and the time interval (Δt) have great influence on the accuracy of computation of the piezometric head and hence on the concentration of wastewater. After number of trials, the numerical calculations using N = 25 and Δt = 0.5 days gave an accuracy for the piezometric head to second figure after the decimal point.

(i) Single Well Results:

Figure 5 shows the variation of the piezometric head with the radial distance from a recharging well for various times. It is obvious that the drawdown decreases with the increase of radial distance r and increases with the increase of time.
The difference of the drawdown values at different time levels are summarized in Table 1. Apparently, the percentage increase of the drawdown becomes constant at nearly 3.5 % at time level greater than 20 days. Therefore, the steady state condition (or stability) of the drawdown curve is reached after time nearly equal 20 days. This is in close agreement with experimental findings [8, 9].

Table 1 Values of piezometric head at radial distance r = 56.7 m with different time levels for Q = 2000 m^3/day, T = 200 m^2/day and k = 50 m/day.

Time (days)	Drawdown	Difference	% Increase
10	4.102	-	-
15	4.440	0.338	8.2
20	4.03	0.163	3.6
25	4.763	0.160	3.4

The analysis is extended to illustrate the variation of the pore velocity (U) with the radial distance (r) from a well for different transmissibility parameters as in Fig. 6. It is clear that the pore velocity decreases as the distance from the well increases. But, the rate of decrease is very rapidly near the well and then becomes slow away from the well. This due to the gradient of the drawdown curve (Fig.5). The pore velocity depends as well on the transmissibility parameter where the pore velocity decreases with the increase of it. The numerical values of U obtained are in range of experimental observations (U= 0.03 to 15 m/day) referred in [14].

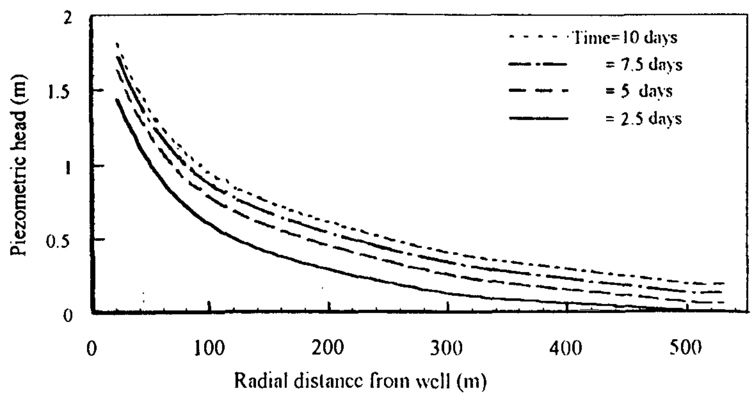


Fig. 5 Piezometric head (h) versus radial distance (r) for different Times

The variation of the relative concentration (C/C_0) with the radial distance is shown plotting in Fig. 7 for different times. As expected, the concentration decreases rapidly away from the recharging well. This attributes to the rapid decrease of the pore velocity with the increase of radial distance from the well. The relative concentration C/C_0 depends largely on the time [9, 11].

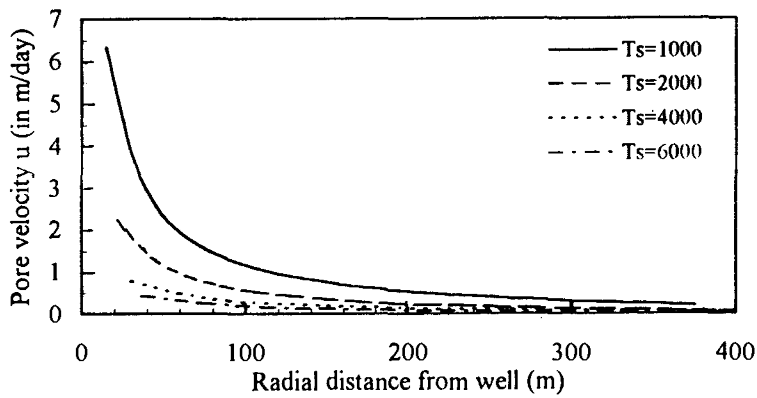


Fig. 6 Pore velocity (U) versus radial distance (r) for different T_s values, k = 50 m/day and $K_t = 5x10^6$).

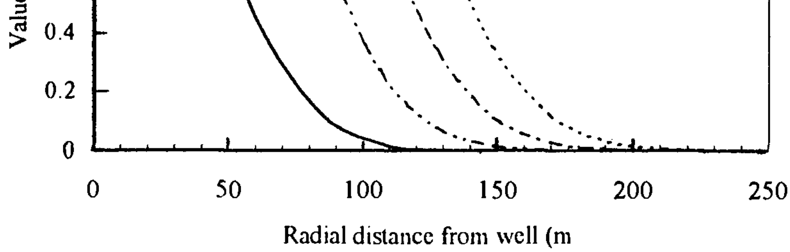


Fig. 7 Relative concentration C/C_o versus radial distance from well with time for ;k= 50 day ,T= 100 m^2/day and T_s= 2000

(ii) Multi-well Results

Fig. 8 illustrates the variation of the piezometric head at point A with time for different T_s values. In this figure, the piezometric head is largely dependent on the transmissibility parameter T_s where h_A decreases with the increase of T_s values.

According to Eq. (20), the relative concentration C_A/C_0 (for $n_w = 5$) is plotted in Fig. 9 versus the time-parameter K_t, for various values of T_s. It is apparent that the C_A/C_0 value depends largely on both K_t and T_s values. The relative concentration increased with the increase of K_t but it is inversely proportional to T_s.

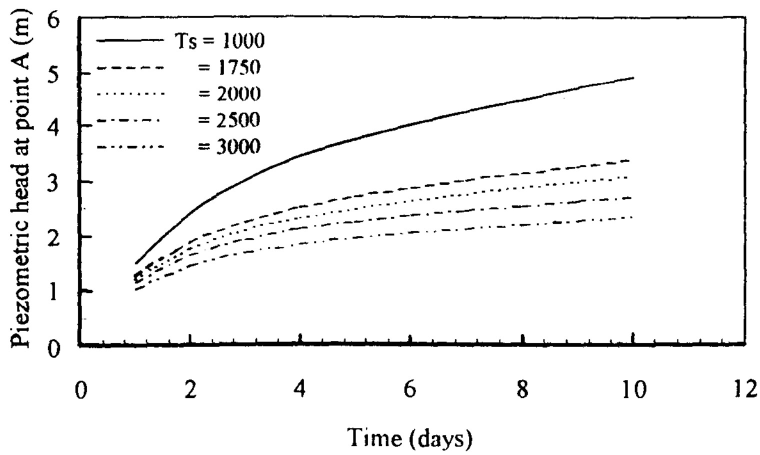


Fig. 8 Piezometric head at A versus time for different values of T_s (k = 100 m/day, d_s = 1.00 mm, φ= 0.4 and S= 0.004).

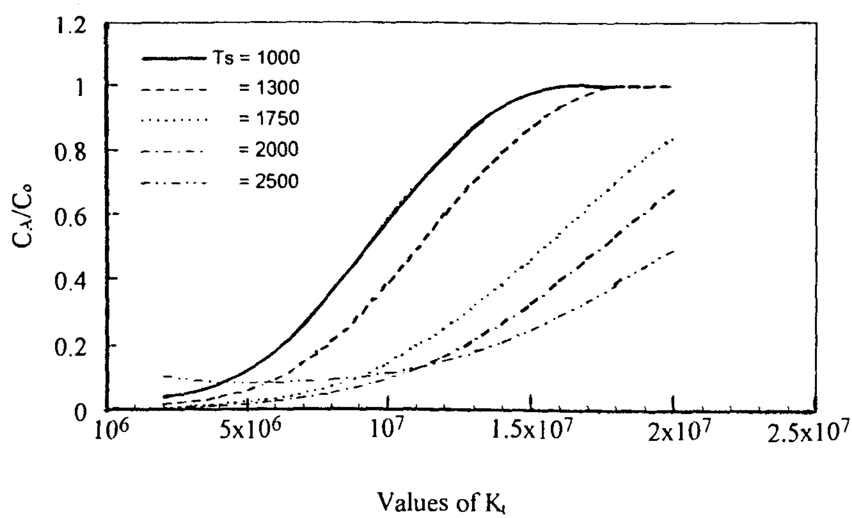


Fig. 9 C_A/C_o versus K_t for various values of T_s for 5 wells.

In similar way, the results of C_A/C_o with K_t and T_s values for number of wells $n_w = 3$ are drawn in Fig. 10. For the same values of K_t and T_s, the C_A/C_o values slightly decreases with the decrease of n_w.

Fig. 11 shows the relative concentration C_A/C_o versus K_t for number of interfered neighboring wells (n_w). Obviously, the value of C_A/C_o increases slightly with the increase of number of interfered wells with maximum percent not exceed than 7%.

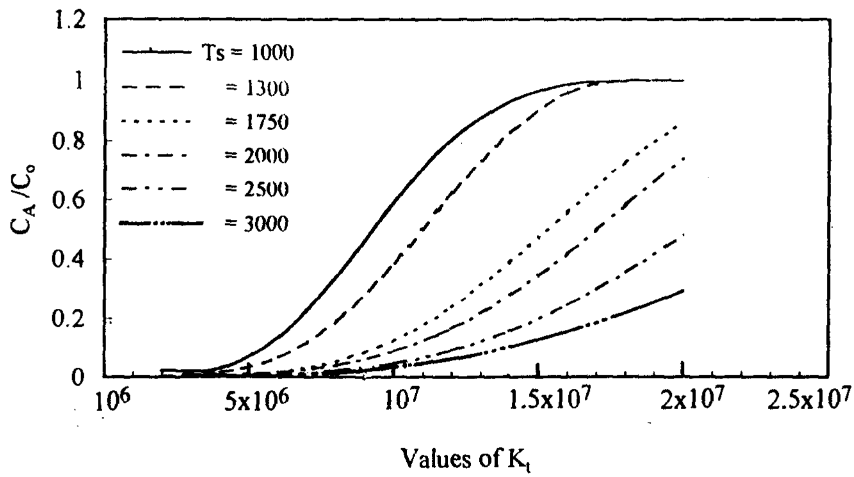


Fig. 10 C_A/C_o versus K_t for various values of T_s (for 3 wells)

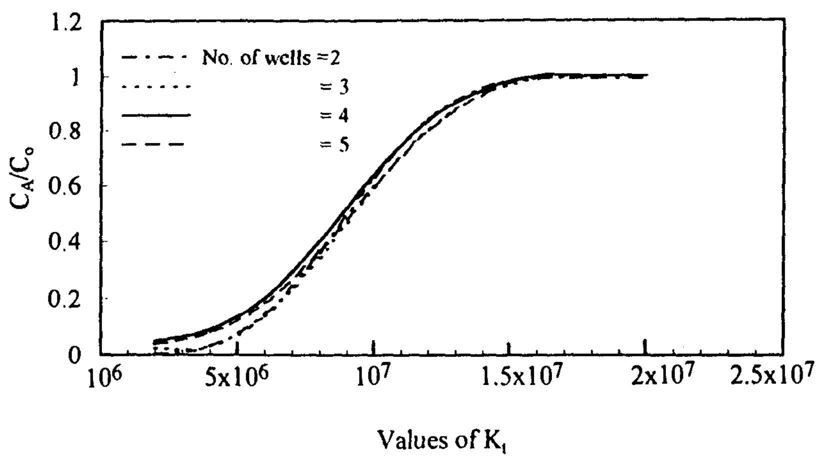


Fig. 11 C_A/C_o versus K_t for different No. of wells at $T_s = 1000$

CONCLUSIONS

The conclusions drawn may be summarized as follows:

1- The finite difference technique is fruitfully used for solution of partial differential equations representing the unsteady flow with the dispersion of waste-water through multiple-recharge wells in porous media.
2- For confined aquifer, the drawdown curve (for recharge well) is dependent on both the radial distance and time. The steady state condition of the drawdown reaches after nearly 20 days.
3- The pore water velocity does depend on the radial distance, the transmissibility-parameter and the gradient of the drawdown curve.
4- Both the time-parameter ($K_t = k\,t/\,d_s$) and transmissibility-parameter ($T_s = T/\,k\,d_s$) influence the relative concentration C_A/C_0 at a point of interfered-recharge wells.
5- The number of recharged wells influences slightly the relative concentration of wastes at a reference position in the field of neighboring recharging wells.

NOMENCLATURE

The letter symbols introduced in this paper are defined where they first appear and are summarized as follows:

A, a = coefficients;
B, b = coefficients;
C_i^j = concentration at radial distance r_i and time level j;
C_o = initial concentration;
D, d = coefficients;
d_s = grain size;
e = coefficient;
F, f = coefficients;
g = coefficient;
h_i^j = piezometric head at distance r_i and time level j;
k = permeability coefficient;
K_r = radial dispersion coefficient;
K_t = dimensionless time-parameter = k.t/ d_s;
N = number of blocks;
n_w = number of wells;
Q = recharge rate of waste-water;
R = radius of influence;
S = storage coefficient;
T = transmissibility coefficient;
t = time;
Δt = time increment;
U = pore velocity and
φ = porosity of soil.

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